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A Cross-Sectional Investigation of the Development of the Function Concept

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ABSTRACT. This study investigates students' development of the function concept as they progress through undergraduate mathematics. An exam measuring students' understandings of major aspects of the function concept was developed and administered to students who had just received A's in college algebra, second-semester honors calculus, or first-year graduate mathematics courses. Follow-up interviews were conducted with five students from each of the three groups. Analyses of the exam results and interview transcripts reveal that even our best students do not completely understand concepts taught in a course, and when confronted with an unfamiliar problem, have difficulty accessing recently taught information.

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The college algebra students had a narrow view of functions and believed that all functions were definable by a single algebraic formula. They did not understand the function language, were unable to interpret dynamic graphical information, and did not know how to use function notation to represent "real world" relationships. Their inability to speak and think about functions as processes which accept input and produce output suggests they conceptualized functions as actions. Second-semester calculus students had a much more general view of functions and much greater ability to speak and use the language of functions. Although they were able to interpret dynamic graphical information, they were unable to use information taught in early calculus and had difficulty interpreting and representing covariant aspects of a function situation. The beginning graduate students understood most aspects of the function concept and had a greater tendency to access concepts of beginning calculus and the ability to represent covariant aspects of a function situation.

These high performing students believed that their mathematical abilities were acquired during high school and were developed as a result of confronting difficult problems in an environment where they were encouraged to reflect, persist, and engage in constructive activities. They wanted to understand newly presented concepts and were frustrated that the rapid pace of particular classes had led them to abandon understanding and retreat to memorization. These results indicate that an individual's view of the function concept evolves over a period of many years and requires an effort of "sense making" to understand and orchestrate individual function components to work in concert.

Introduction

The function concept is an important and unifying concept in modern mathematics [9, 5, 18], central to many different branches of mathematics [8], and essential to related areas of the sciences [18]. A strong understanding of the concept of function is a vital part of the background of any student hoping to comprehend calculus [2]. Beyond their use in calculus and analysis, functions are widely used in the comparison of abstract mathematical structures. For example, functions are used to determine whether two sets have the same cardinality and whether two topologies are homeomorphic. Functions can also be used as elements of abstract mathematical structures such as vector spaces, rings, and groups. Further, functions are used extensively in the sciences for modeling such phenomena as brain activity, population density, and electrical fields. Despite the fact that functions are currently recognized as a unifying mathematical concept and an important mathematical construct, Cooney and Wilson's [4] historical investigation of the function curriculum demonstrates that textbook authors have not made the function concept a unifying principle of early algebra curriculum.

As early as 1921, the National Committee on Mathematical Requirements of the Mathematical Association of America recommended that the study of functions be given central focus in secondary school mathematics [4]. The NCTM *Curriculum and Evaluation Standards* [15] calls for the inclusion of function-related activities as early as fourth grade (p. 60), continuing through the high school mathematics curriculum where the concept of function is a unifying idea (p. 154). In *Everybody Counts* [16], the authors state, "if undergraduate mathematics does nothing else, it should help students develop function sense" (p. 51). Although curriculum reform efforts are beginning to respond to calls for change, and researchers have begun to identify many of the difficulties students experience with understanding aspects of the function concept [6], ongoing analysis of student understanding of the function concept is necessary for guiding future curriculum decisions. Without such analysis, curriculum decisions stand little chance of being guided by informed judgments of how students acquire understanding of essential function components.

This study investigates how high-performing undergraduate students acquire an understanding of major aspects of the function concept. A 25-item exam was developed and administered to students who have just received A's in college algebra, second-semester calculus, or beginning graduate mathematics courses. The written exam was designed to investigate the following research questions:

- What differences exist among the three groups relative to their understanding of major aspects of the function concept?
- When are students' understandings of major aspects of the function concept acquired?

Following the administration of the written exam, follow-up interviews were conducted with five students from each of the three groups. The follow-up interviews were designed to investigate the following research questions:

- What factors in students' backgrounds contribute to existing differences among the three groups relative to their understandings of major aspects of the function concept?
- What factors in high-performing students' backgrounds have influenced their mathematical development and continued study of mathematics?

Although researchers have investigated many different aspects of students' understandings of the concept of function, such as: students' ability to interpret various types of graphs [24]; students' conceptual view of functions [2]; and students' ability to translate between various function representations [5, 21, 19], a comprehensive study aimed at guiding educational reformers has not yet been devised. To make this study more inclusive, research investigating many aspects of the function concept which previously had been considered separately [2, 14, 5] was brought together. This research investigates students' abilities to:

- Characterize "real world" functional relationships using function notation;
- Operate with a particular type of function representation, such as a formula, a table, or a graph;
- Move between different representations of the same function;
- Represent and interpret covariant aspects of the function situation (i.e. recognize and characterize how change in one variable affects change in another);
- Interpret "static" and "dynamic" functional information (i.e. interpret graphs representing position and rate of change);
- Interpret and describe local and global function properties: slope, continuity, and differentiability;
- Construct functions using formulas and other functions;
- Recognize functions, non-functions and function types;
- Conceptualize a function both as a process and as an object;
- Interpret and understand the language of functions; and
- Characterize the relationship between a function and an equation.

This list of abilities provides a framework for investigating changes in students' function conceptions. It takes into account Breidenbach et al.'s [2] and Monk's [13] classifications of students' conceptual views and additional aspects of a mature function understanding which were identified by this study. As students gain a more complete understanding of functions they acquire more of these abilities. This list of function abilities was developed and refined over a four-year period, each one measurable on multiple levels and frequently separated by fine distinctions. This framework provides flexibility in the investigation of relationships among students' function abilities and conceptual views. For example, a student's function conception can be related to her or his ability to represent covariant aspects of a function situation and/or ability to interpret graphs representing rates of change.

Background

Research results show that acquisition of essential aspects of the function concept is extremely complex [18]. Students have difficulty translating between different representations [19] and applying basic concepts at different levels of abstraction [9]. Ayers et al. [1] and Vinner and Dreyfus [21] report that many students think a function must only be represented by a single algebraic rule describing a continuous, one-to-one function. Selden and Selden [18] also report that students think the graphs of functions should be "nice," and all functions must be one-to-one.

Monk [11, 12, 13] has done extensive research investigating students' interpretations of graphs. His recent research [13] has shown that students often experience problems interpreting dynamic graphical relationships over subintervals of the domain of a function. Monk [13] and Kaput [7] both report that students expect the shape of a graph to reflect visual aspects of the situation described by the graph, rather than a representation of the relationship between two variables.

Educational researchers have classified students' understandings of functions according to their conceptual views. Breidenbach et al. [2] have categorized students' function conceptions as prefunction, action, process, and object. They suggest that early function curricula should aim toward moving students from the action conceptualization, a view of the function as a repeatable mental or physical manipulation of objects, to that of a process conceptualization, the interiorization of actions so that all actions can take place entirely in the mind of the subject. Students attaining this level of understanding will view an expression as "what you get" when you evaluate it says Thompson [20], and will have no difficulty understanding function composition or the relationship between a function and its inverse [2]. Once the student attains a high degree of awareness of a process in its totality, this process can be encapsulated to obtain an object conception [2].

Subjects, Procedures and Data Analysis

The student subjects for the study were selected from three different levels of mathematical preparation: college algebra, second semester calculus, and beginning graduate study in mathematics. Group 1, comprising 30 students just completing college algebra with a high A (greater than 95% class average), was the lowest level group participating in the study. Group 2 consisted of 16 students just completing second semester calculus with a grade of A, and group 3 contained 14 graduate students who had just completed their first semester of graduate level mathematics with a grade of A in either complex analysis or abstract algebra. The college algebra curriculum involved an early introduction of functions in small lecture sections, and the calculus curriculum was taught using a traditional text with lecture also as the primary mode of instruction. The graduate students were products of undergraduate and graduate traditional mathematics curricula. The written exam was administered to each group upon completion of their respective courses. Prior to distributing the written exam, the author announced a monetary incentive for completing the exam with diligence, dedicating at least one and one half hours to the task and showing a serious effort in responding to each item. Exams were scored using a carefully developed and tested five-point rubric for each exam question. After scoring each exam, group means and standard deviations for each group on all exam items were computed. Group differences relative to each question were determined using an F-test, with follow-up pairwise comparisons using the Tukey test at an overall α level of .05.

Follow-up interviews were conducted with fifteen students, five from each of the three groups. Interview subjects were selected by identifying students within each group who performed at various levels on the written exam. Invitations to participate in the interview were given to a broad group with diverse performance in order to have the best representative sampling available. Although the interviews for this study were primarily unstructured, with the interviewer spontaneously reacting to students' descriptions of their solutions, some structure was imposed by preparing interview questions in advance. During the interview, the researcher initially read each exam question aloud and made general reference to the response the student had given. The student was given a few minutes to review the response, then prompted to describe her or his solution and asked to provide justification and clarification to the solution offered. After the student's summary, the researcher made general inquiries, such as, "explain" or "clarify," and continued to ask more specific questions, if necessary, until a response was elicited or it appeared that all knowledge had been elaborated. If one of the main components of the question was correctly answered, the student was queried to recall when and how the concept was acquired. This process was repeated for each question on the written exam.

Interview length varied from 90 to 150 minutes. Group 1 students expended the least amount of time due to their limited ability to articulate responses to some of the more difficult exam items. During the interview, correct responses were acknowledged and students were encouraged to communicate their ideas both verbally and symbolically. The interview tone was amiable and nonthreatening, and efforts were made to make students comfortable with providing candid responses.

Analysis of the interview results involved a careful reading of each interview transcript, while attempting to identify common student responses and misconceptions. The percentage of students providing each response type was then determined for the collection of interviews for each group.

During interviews, students were also asked to describe what they believed had affected the development of their mathematical abilities, as well as their interest in continuing to study mathematics. These sections of the interview transcripts were analyzed separately to provide information regarding the factors that influence successful students' mathematical maturation and motivation. Common responses were identified and the percentages of students providing each response type, for each group, were determined.

Final results were obtained by analyzing both the quantitative and qualitative results relative to each group, as well as the individual.

Development of the Research Instruments

The written exam was designed to measure major aspects of acquiring an understanding of the function concept. The exam development paralleled the identification of these attributes. Different exam problems sometimes measure different levels of understanding of the same function attribute, and one problem frequently provides information about more than one function ability. Many of the exam items were constructed in collaboration with experts in the field; and others were borrowed from the literature. For example, Monk's "speed vs. time for two cars" problem [14] was used to measure students' abilities to interpret static and dynamic graphical information, and selected items from Breidenbach et al.'s [2] exam measured students' function conceptualizations.

Five-point rubrics were written for each exam question. Each rubric was developed to measure the accuracy, strength of justification and degree of conceptual understanding shown by responses. Prior to developing the individual rubrics, a general rubric guideline combining aspects of both the Kansas and California rubrics used in state mathematics testing, was devised to define criteria for determining individual points for specific rubrics.

Development and verification of the 25 rubrics involved a lengthy process of refinement. Following the development and refinement of individual rubrics, four experts were solicited to review all rubrics, noting inconsistencies between their opinions and the developed rubrics. There were few such inconsistencies, though minor refinements were made using the experts' feedback. After scoring each exam twice, resolving all inconsistencies between the two scorings, and again making minor refinements to individual rubrics, the reliability of the rubrics was verified by asking each expert to score one exam from each group. The average variation between scores was 3.5% and the greatest variation between two scores was 4.8%. No score on an individual test item varied more than one point out of five. Since the experts were familiar with the problems and had been working closely with the researcher during the development of the exam, six additional individuals, three mathematicians and three mathematics educators, were solicited to score three exams, again one from each group. With this scoring, the average variation between scores was 5.5% and the greatest variation was 7.2%.

The research interview was designed to determine the knowledge motivating written responses and gain additional information concerning how and when particular constructs were acquired. Its development was guided by careful examination of each written exam response. During an initial reading of all exams, common responses for individual items for each group were identified. The exams were read once again, while tallying the number of students within each group who provided each of the response types. Prior to conducting each interview, the interviewee's exam was re-read, and notes were made concerning evidence of incomplete understanding. This information and the tallied common responses provided guidance for developing interview questions and focus for conducting individual interviews.

Results

Because of the large amount of data collected, details are presented only for selected exam items. Rubric scores for all items are in Appendix B. For a complete description of results for the 25-item exam for all groups, see Carlson [3].

The presentation of results includes a statement of the question, a brief discussion of the results, and a table presenting exam scores and common written responses for each of the three groups. This is followed by a discussion of the interview results of the group which provided the most interesting interview results for that item, followed by excerpts of individual interview transcripts for each interview subject within the group.

Item 5. (Instructions for items 5–7: Give an example to confirm the existence of such a function. If one does not exist, explain why.)

Does there exist a function all of whose values are equal to each other?

Groups 2 and 3 performed relatively well on this item (Table 1), with group 1 having a mean score of 1.07 (out of 5.0). Both groups 2 and 3 performed significantly higher than group 1, while group 3 did not perform significantly higher than group 2. A correct example was constructed by only 7% of group 1 students, while 69% of group 2 and 92% of group 3 students provided a correct example. Twenty-five percent of group 2 students constructed the example $y = x$ and 8% of group 3 students constructed an example containing minor errors.

TABLE 1. Quantitative Results for Item 5

	Group		
	1	2	3
Mean score[a]	1.07	4.00	4.67
Standard deviation	1.76	1.87	1.15
Common responses (%)			
No answer	60	0	0
$y = x$	23	25	0
Minor error in example	10	6	8
Correct example	7	69	92

[a] The differences between the means of groups 1 and 2, and groups 1 and 3 were significant at $\alpha = .05$.

Four of the five students interviewed from group 1 persisted with "$y = x$" as the answer, when asked to construct a function all of whose values are equal to each other. The interviews for this question suggest that group 1 students do not understand that the function values represent the y-values (assuming traditional labeling). When prompted to explain what is meant by the phrase, "all of whose values are equal to each other," two students gave a response indicating that all the x-values are equal (students B and E), and two students indicated that all x-values must equal all y-values (students A and C). The interview transcripts for group 1 students revealed that high-performing college algebra students are not able to translate verbal function language to algebraic function notation for a basic, but essential, aspect of functions.

Interview Transcripts—Group 1.

Student A.

Int: Can you define a function all of whose values are equal to each other?

A: I really cannot compute anything on that. Its values are equal? I really do not know. Is that a function?

Int: Would $y = 5$ work?

A: I guess it is equal, but I do not know.

Int: Do you think it meets the criteria, a function all of whose values are equal to each other?

A: I guess, not really, because $y =$ anything that is not $y = 5$.

Int: So when you hear, "all of whose values are equal to each other," what do you think of?

A: I was just thinking when x equals to y, or two different variables equal to each other.

Student B.

Int: You said yes, but did not give an example. Can you think of one?

B: I just thought about the one at $\{(0,0),(1,1),(2,2)\}$. I don't know if that is right.

Int: How would you write that function?

B: Like $x = 1$, $y = 1$. So at $(0,0)$, $(1,1)$. Is that what you were thinking of?

Int: What am I referring to when I ask, "whose values are equal to each other"?

B: Like as if x values are always equal. Then it would be a vertical line.

Student C.

Int: You said yes, and you constructed the function $y = x$. Explain to me why you think this function works.

C: I was thinking all x values are equal to all y values, like if you got one side of the equation, a number is the same number on the other side of the equation. That would fill in the whole chalk board. No, it would not fill in everything, it would be a line.

Int: Does this meet the criteria that all values are equal to each other?

C: A function all whose values are equal. No, because what you want is something where y equals and all of the x would be equal, so you want x

to be the same. That will be a straight line, a vertical line. But it would not be a function, because you cannot have a vertical line as a function, because it would not pass the vertical line test.

Int: How did you determine that you wanted all of the x values equal to each other?

C: y is just arbitrary. I mean, not arbitrary but a solution. If you want all values equal to each other, then the x values that you plug in the formula are equal to each other. So all x need to be the same.

Int: It looks like you are still thinking.

C: Well, there is something that I am not quite grasping.

Student D.

Int: You left this blank. Can you think of a function all of whose values are equal to each other. What do you think I'm asking you to find?

D: I would just think that no matter what value you put in there, you have to get the same thing. I guess, if you have a function $f(x) = x^0$, that would work.

Int: Excellent! How about $f(x) = 7$, would that work?

D: Yes. In that one, there is no variable there, so there will be no change. I was thinking you had to have a variable.

Student E.

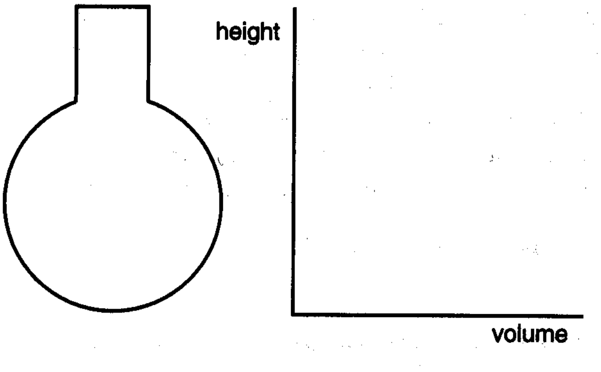
Int: You said yes, and wrote absolute value and negative absolute value. Can you explain your answer?

E: I just think of the graph when $x = y$. I mean absolute value. I was thinking of this graph of a straight line. They are all equal to each other. When $x = 2$, $y = 2$.

Int: So, how do you interpret, "whose values are equal to each other"?

E: All x's equal all y's.

Item 12. Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle.



The mean scores for the three groups were significantly different for item 12 (Table 2), with group 3 performing higher than both groups 2 and 1, and group 2 performing significantly higher than group 1. Group 2's scores are more variable than those for group 1 and 3, with group 2 receiving a mean score of 2.41. Analysis of the written exam revealed that 74% of the group 2 subjects constructed a strictly concave down or concave up graph for the portion of the graph corresponding to the spherical portion of the bottle.

TABLE 2. Quantitative Results for Item 12

	Group		
	1	2	3
Mean score[a]	1.47	2.41	4.33
Standard deviation	1.31	1.46	0.89
Common responses (%)			
Constructed an increasing straight line	47		
Another incorrect graph	26		
Constructed a strictly concave up graph	27	56	14
Constructed a strictly concave down graph		18	14
Correct graph, except for slope of line		13	21
All aspects of graph were correct		13	51

[a] The differences between the means of groups 1 and 2, and groups 1 and 3, and groups 2 and 3, were significant at $\alpha = .05$.

Given a bottle filling with water and asked to express height of water in the bottle as a function of volume, group 2 interview results reveal very different levels of understanding of covariant aspects (i.e. how change in height affects change in volume) of this real world situation. One of the five students (L) provided a strong justification, which clearly demonstrated understanding of the physical situation in terms of calculus. He provided a clear description of the concavity change in terms of the rate of change, and fluently discussed the changing sign of the second derivative. Two students (J and N) provided a clear analysis of the situation, using algebraic reasoning. Student M demonstrated no ability to attend to the covariant aspects of the situation. He simply argued that the height would always be increasing, so its graph would be a straight line. Student K provided a strictly concave down graph. Although this student demonstrated some awareness of the changing rate of height with respect to volume, he concluded the interview supporting the incorrect graph.

Interview transcripts—Group 2.

Student J.

Int: Describe how you sketched the graph (graph is correct).

J: I knew it changed different for the bottom part because it's circular and the top part has straight walls. If you look at it as putting the same amount of water in each time and look at how much the height would change, that's basically what I was trying to do. So for the first part, the height would be changing more quickly, and in the middle if you add the same amount of water, the height would not change as much as it would at the bottom. It's symmetric.

Int: How does that affect the graph?

J: Higher slope in the beginning, then it levels out, then a higher slope again. Then for the neck part it's basically a straight line because you're filling the same area with each amount.

Int: Nice explanation. What is the slope of the straight line?

J: About like the curve right here (pointing to the junction of the curve and line).

Student K.

Int: Explain your solution (graph is concave down).

K: This is my least favorite problem. I tried to solve for height in terms of volume and it was a mess.

Int: Can you analyze the situation without explicitly solving for h?

K: OK, the more water, the higher the height would be. In terms of height of the water, that is what we are talking about. If you are talking about the height left over, that is basically decreasing. Right here the height will be zero and the volume is zero. As you go up a little more, height increases and the volume increases quite a bit. Once you get there, the height increases slower. I guess from here to there height increases the same as the volume increases, and once you get here it increases slower. I guess from here to there height increases the same as the volume increases. No, I am wrong again. So, every time you have to put more and more volume in to get a greater height towards the middle of the bottle. Then once you cross the middle point of the bottle you have to put less and less in to get a greater height. Once you get here, it would be linear, probably. So, it's always going up (he traces his finger along the concave down graph), then it would be a line.

Int: So, what does the graph look like?

K: Like this (pointing to the concave down graph he has constructed), but it has a straight line at the end.

Student L.

Int: Explain how you obtained your graph (correct graph).

L: Well, looking at the volume of the sphere, if you take that, I remember my units have to be cubic, so I know it's some sort of cubic equation. For this cylinder part, I know it's going to be linear, since for the cylinder it's related by volume which equals area times height. And so we have area as a constant. So what we have is a linear equation for height as it's related to volume. Getting to that point, I knew it was filling at a cubic rate somehow, so it would have something like a cubic equation. When you take the inverse of that equation it whips it like that. But I was also able to see here that when you start out, it's going to be filling rapidly, so you are going to have a great slope. (Some confusion and continues.) But as you increase the volume, you're going to get less of a height change until you get up to here. As you get past the half-way point, it's going to go from concave down to concave up and you're going to have an inflection point.

Int: Can you tell me why it changes concavity there (pointing at the inflection point)?

L: Because if you take the second derivative of this volume in terms of height, you'll get a zero. On this side you have a negative acceleration. But once you reach the half-way point, then you start becoming a positive second derivative.

Int: Good.

Student M.

Int: Can you explain your solution (graph is concave up)?

M: I tried to solve for h. But I think I need to define it as a piecewise defined function. Maybe then I can figure it out.

Int: Did you try to get an idea of the general shape of the graph by imagining the bottle filling with water?

M: As the volume comes up, the height would go up at a steady rate.

Int: How would you represent this graphically?

M: It would be a straight line.

Int: So, the entire graph is a straight line.

M: Yes.

Student N.

Int: Can you explain how you determined your graph (graph is correct, except for the slope of the linear portion)?

N: When you're given a flask like this, the way I thought of it was, you have to start the coordinates at $(0,0)$ with volume equal to 0, and the height equal to 0. When you start filling something that has such a wide base like this, the height is going to not increase as fast as the volume, so that's why I have more of a line which is sloped more, like a little less than 1. Then when it gets past the circle, the largest diameter, here is when it is going to switch to the height increasing faster than the volume and thus the slope is going to be greater than 1 and increase that much quicker.

Int: What happens when it gets back to the neck?

N: The height will increase unit-wise compared to the volume, so it should be a straight line, I guess.

Int: How does the slope of the straight line correspond to the slope of the graph where the linear portion begins?

N: The slope would be just a little bit greater. It would be more like it was right where it went into that neck.

Item 4a. (Part B) Tom sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount and then again by the same amount and then again by the same amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do the amounts by which the top of the ladder drop down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? EXPLAIN.

The mean scores for the three groups are significantly different for item 4a, part B of the written exam (Table 3), with group 3 performing significantly higher than both groups 1 and 2, and group 2 performing significantly higher than group 1. Both quantitative and qualitative results indicate that group 2 and 3 students had very different levels of understanding and employed many different approaches to solving this problem. Group 1 responses were weak, with 67% of the written exam responses indicating that the ladder either remains constant or gets smaller. Twenty-three percent of group 2 students provided the correct solution, with an algebraic justification, and 29% provided the correct answer, but did not provide any explanation for their answers. Ninety-one percent of group 3 students provided the correct response, with 58% providing an algebraic justification and 33% using calculus to justify their answers.

TABLE 3. Quantitative Results for Item 4a (Part B)

	Group		
	1	2	3
Mean score[a]	0.63	2.41	4.33
Standard deviation	1.03	1.46	1.61

[a] The differences between the means of groups 1 and 2, and groups 1 and 3, and groups 2 and 3, were significant at $\alpha = .05$.

Group 1 students were unable to represent the position of the ladder on the wall using algebraic techniques, though two of the five interview subjects attempted to reason through a solution by constructing their own physical models with their pencils. Group 2 students' responses varied dramatically on this item, with each interview subject providing a different approach. Individual responses for group 2 students ranged from totally correct with a clear justification using calculus to a response that the ladder drops down by a constant amount, followed by an incorrect algebraic argument. Analysis of group 2 interview results reveals that when analyzing real-world relationships second-semester calculus students do not demonstrate the tendency to access the formal mathematics that they know. However, four of the five interview subjects for group 3, without hesitation, set up the Pythagorean relationship representing the position of the ladder on the wall. Three of these students finished their solutions by computing the derivative, followed by a correct justification using the language of calculus. The other student provided a correct algebraic justification. The only group 3 student unable to provide a correct solution had just returned to graduate school and had not interacted with early calculus for over 8 years. The students who accessed tools of calculus when justifying their responses had recently taught a course in introductory calculus.

Item 2d. Compute $f(x+a)$ given $f(x) = 3x^2 + 2x - 4$.

Group 2 and 3 students demonstrated little difficulty (Table 4) with this item. However, group 1 students' mean score of 2.07 is surprising, since their college algebra course had provided explicit instruction in algebraically computing $f(x+a)$ for both linear and quadratic functions. The most common incorrect written exam response (43% of group 1 students) was to simply add an a to the expression on the right side of the equal sign. Group 2 and 3 students had no difficulty with this item, although one group 2 student failed to complete the question.

TABLE 4. Quantitative Results for Item 2d

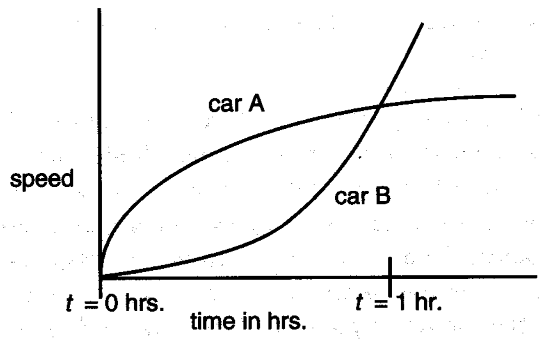
	Group		
	1	2	3
Mean score[a]	2.07	4.76	5.00
Standard deviation	2.32	0.97	0.00

[a] The differences between the means of groups 1 and 2, and groups 1 and 3, were significant at $\alpha = .05$.

Although four of the five group 1 interview subjects provided correct justifications for their correct responses, their justifications provided insights into how group 1 students think about the evaluation of $f(x+a)$. Group 1 students described their solutions either as a substitution of $x+a$ for x, or a procedure of adding a to every x. When prompted for a more in depth explanation, none of the interview subjects referred to the fact that they were evaluating f at another point, or the input to f was now $x+a$, rather than x. The student who simply added a to the expression on the right of the equal sign indicated that he had arrived at this solution by substituting $3x^2 + 2x - 4$ into the x in $f(x+a)$. Additionally, students C and E stated that they once had difficulty with this type of problem because they did not know "which one to plug into which one." Although no interview subject viewed the problem as simply adding a to both sides of the equal sign, this justification did occur when piloting the interview procedures. These responses suggest that group 1 students view function evaluations as collections of actions to be carried out, and do not yet view a function as a process which accepts input and produces output.

In their interviews, group 2 and 3 students either discussed their correct solutions by describing $x+a$ as the input of the function, or indicated that they were evaluating the function f at $x+a$, and that the solution after the evaluation was the output to the function. These responses suggest that group 2 and 3 students have a process view of functions (i.e. they think of a function as a process which accepts input and produces output).

Item 8a. The given graph represents speed vs. time for two cars.



State the relationship between the position of car A and car B at $t = 1$ hr. (assume the two cars start from the same position and are traveling in the same direction). Explain.

Item 8d. What is the relative position of the two cars during the time interval between $t = .75$ hr. and $t = 1$ hr.? (i.e. is one car pulling away from the other?)

Results—8a.

On item 8a, group 3 with a mean score of 5.0, performed significantly higher than groups 2 and 1. Group 2, with variable scores (Table 5), performed significantly higher than group 1. Forty-seven percent of group 1 students stated that the cars were in the same position and 34% responded that car B was passing car A. These responses suggest that 88% of group 1 students interpreted the graphs literally as the paths of the cars, rather than interpreting the functional information displayed by the graphs. Somewhat surprisingly, 29% of group 2 students also demonstrated this misconception. However, all group 3 students provided the correct response with 42% justifying their answer by comparing the relative areas under the curves.

Two group 1 interview subjects indicated that the cars are colliding at $t = 1$ hr., since their paths are intersecting, and another student stated that the cars are moving away from one another at $t = 1$ hr. The remaining two subjects pro-

TABLE 5. Quantitative Results for Item 8a

	Group 1	Group 2	Group 3
Mean score[a]	0.83	2.82	5.00
Standard deviation	1.74	2.48	0.00
Common responses (%)			
The two cars are at the same position	47	29	0
Car B is passing car A	34	0	0
Correct solution, algebraic justification	19	53	58
Correct solution, used calculus to justify	0	18	42

[a] The differences between the means of groups 1 and 2, groups 1 and 3, and groups 2 and 3 were significant at $\alpha = .05$.

vided a correct explanation, justifying that car A was ahead of car B since it had been traveling faster for the entire time. Although attempts were made during the interviews to redirect the students' attention to the information displayed by the graphs, none of the students who provided incorrect responses attempted to rethink the problem. They all continued to provide responses suggesting they were interpreting the graphs as the paths of the cars. Group 2 interview subjects all provided the correct response, stating that car A is ahead of car B, with two subjects comparing the relative areas under the curves, and three subjects comparing the relative speeds of the two cars. Four of the group 3 interview subjects justified their correct responses by comparing the relative areas under the curves. The remaining student provided an algebraic justification comparing the relative speeds of the two cars.

Results—8d.

Group 3 performed significantly higher than group 2, but group 2 with a mean score of 1.18, did not perform significantly higher than group 1. The majority of group 1 students provided an incorrect response to this item (Table 6). In fact, 43% of the written exams for group 1 indicated that car B is catching up to car A, and 33% indicated that the cars are "moving toward" or "crashing in" to each other. Groups 2 and 3 also had difficulty with this item, with 63% of group 2 and 42% of group 3 students stating that car B is catching up to car A.

Each group 2 interview subject justified the incorrect response (car B is catching up to car A), by stating that greater acceleration means "catching up." These students all demonstrated heavy reliance on this incorrect information, rather than relying on the information displayed by the graph. In addition, only three of the five group 3 interview subjects gave the correct response, providing the justification that car A is traveling faster the entire time. The remaining two group 3 interviewees persisted with incorrect arguments, one stating, like group 2 interview subjects, that acceleration equates with "catching up," and the other

TABLE 6. Quantitative Results for Item 8d

	Group		
	1	2	3
Mean score[a]	0.43	1.18	3.83
Standard deviation	1.17	1.38	1.27
Common responses (%)			
Cars are moving in to each other/crashing	33	12	
Car B is catching up to car A	43	63	42
Another incorrect response or no response	14		
Car A is pulling away from car B	10	25	58

[a] The differences between the means of groups 1 and 3, and groups 2 and 3, were significant at $\alpha = .05$.

indicating car B is getting closer since the difference in the areas under the curves is getting smaller.

Interview transcripts—Group 2.

Student J.

Int: Why do you say car B is catching up to A?

J: A is ahead of B, but B is accelerating more quickly than A, so even if they stay at the same velocity, B is eventually going to catch up and pass A.

Int: Where does B pass A?

J: A good guess would be at one hour I guess.

Int: So, if one car is accelerating more than another, that car will be catching up?

J: Yes.

Int: Look at their relative speeds again. Which car is going faster for the time interval between $t = .75$ hr. and $t = 1$ hr.?

J: Car A.

Int: Does this give you any additional information for answering this question?

J: It will be ahead of car B, but while it's going faster it's not going to keep up with car B.

Student K.

Int: You said car B will be catching up. Can you explain your answer?

K: Car B is going to be catching up because car A is slowing down. Car A's acceleration is not as great as car B's acceleration and so the distance is becoming less and less between the two.

Int: Because car B is accelerating more?

K: Because if you pick a time, say $t = .5$ hr., then car A is still going quite a bit faster than car B, and average speed of A is still greater, and so there is going to be a certain distance between them. And if you pick $t = .75$

hr., car A's speed, even though it is greater than car B's; still it is not as much greater because of the acceleration. So that means distance must be decreasing. So car B is catching up to car A.

Student L.

Int: You said car B is gaining on car A. Can you explain this?

L: Car B is behind car A at that point. But car B is rapidly gaining on Car A because car A is pretty much done gaining speed and is just moving along. Car B is still going faster. At the start, car A had more start off power.

Int: Your response is that car B is gaining on car A?

L: I guess if you wanted to you could say that car B is still dropping behind, but not as fast as it used to be. It's not being left behind. Car A is still pulling ahead though, but not in each time increment as much as it did in the previous. So in that sense car B is gaining on car A.

Student M.

Int: You said car B is gaining on car A. Can you explain this?

M: Car B is gaining on car A because its acceleration was greater.

Int: So, you say car B is gaining?

M: I think the tangent line is the secret here.

Int: Why?

M: It compares the accelerations.

Int: Why did you choose to compare the accelerations rather than the speeds?

M: I guess acceleration simply describes the system better.

Student N.

Int: You said car B makes up ground on car A because it starts to accelerate. Can you explain this?

N: It's gaining because it's accelerating more. Car B increased more than car A right before one hour, Since its accelerating quicker, you know that car B is making up ground on car A. If its accelerating quicker, you know that its changing speed faster, thus its going to make up more distance in that interval.

Int: What do you mean by make up more distance?

N: Well, Car A is ahead of car B at $t = .75$ hr. by so much, and car B decreases that amount in that last interval because it's accelerating.

Int: Because it's accelerating it's closing the gap?

N: Yes, it's closing the gap.

Int: Why don't you consider the relative speeds in that hour?

N: Well, car A is moving faster than car B is, car B's curve, I don't know what you....(trailing off, mumbling).

Int: So you say?

N: Car B is moving faster relative to car A based on what I told you about the acceleration closing the gap.

Int: So you say car A is going faster than car B. Try to use that information to answer this question, keeping in mind that car A is going faster than car B.

N: The relative position would be that car A is always ahead.

Int: If it's going faster, would it be pulling ahead or would car B be catching up?

N: Well just because it's going faster, I would still say that car B is catching up because it's accelerating at a rate where its change in speed is quicker than the change in speed of car A and this is why car A will still be ahead, but car B is still closing the gap.

Item 1. Express the diameter of a circle as a function of its area and sketch its graph.

On item 1, the mean score for group 1 is 1.00, with group 2 scoring significantly higher than group 1 (Table 7). Group 3, with a nearly perfect score, also scored significantly higher than both groups 2 and 1. On this item, group 2 and 3 students had no difficulty providing the algebraic representation of the function. However, students in both groups either neglected to construct the graph, or provided an incorrect graph.

TABLE 7. Quantitative Results for Item 1

	Group		
	1	2	3
Mean score[a]	1.00	3.29	4.83
Standard deviation	1.08	1.10	0.39

[a] The differences between the means of groups 1 and 2, and groups 1 and 3, and groups 2 and 3, were significant at $\alpha = .05$.

When asked to express the diameter of a circle as a function of its area, the interview responses for students in group 1 varied, with only one student providing a correct algebraic response, supported by a well-formulated verbal justification. Interview transcripts indicate that two of the interview subjects did not know the formula for the area of a circle, and when provided with the formula made no attempt to solve for the diameter d. The other two students knew the area formula and successfully substituted $d/2$ for r, but did not attempt to isolate d. During the interviews, each student who provided incorrect responses was asked as a follow-up question to explain what is meant by the statement, "express s as a function of t." One student responded that this statement means that you are trying to find where s and t are equal. Another student responded, "find the zeros," and another said, "kind of how it related to." These responses suggest that group 1 students are unable to translate a verbal function description into algebraic function representation (i.e. they do not know what it means to express one quantity as a function of another). During the interviews, group 2 and 3 students had no difficulty providing a correct response and justification for this item.

Students' Backgrounds and Beliefs

In addition to investigating students' understanding of the function concept, at the beginning of each interview, students were asked to describe their mathematical backgrounds, including any factors which contributed to their mathematical development. Since the group 3 interview results were most revealing and provided interesting insights regarding the background and beliefs of mathematically successful students, a discussion of group 3 interview data is presented and contrasted with those of groups 1 and 2. This discussion is followed by a representative group 3 interview transcript.

Group 3 Interview Subjects.

Group 3 interview transcripts reveal that group 3 interview subjects all possess attitudes, habits and beliefs which equip them to approach mathematics with maturity not found in most members of the other two groups. In addition, the author observed that, while each group 3 subject struggled with a complex task during the interview, he or she exhibited exceptional persistence and confidence. As reported by the students in the interview, their persistence was motivated by different factors: student T, his competitive personality; student U, her curiosity and joy of doing mathematics; student S, her trained discipline instilled by a high school math teacher; student R, her desire to do well and gain control of the many things she had forgotten; student Q, his genuine interest in mathematics and solving difficult problems. In contrast, with the exception of one group 2 interview subject, all group 1 and 2 interview subjects were reluctant to persist, demonstrating little or no confidence in their abilities.

During the interviews, each group 3 interviewee also reported a willingness to dedicate remarkably large amounts of time when attempting to solve challenging problems. Student R, when asked what had contributed most to her mathematical success responded, "I do above and beyond anything that is required. I do the assigned problems and then try some of the harder problems on my own." Student T, when asked if he works hard in his math classes responded, "Yes, I do now, I always try to figure things out if they don't make sense. I put in whatever amount of time that it takes to understand things."

Group 3 interview subjects also indicated that they were influenced by an individual who guided them in learning how to approach mathematics. For student U, it was a high school teacher who made math interesting and taught her how to think mathematically, study for exams, and read mathematical texts (see interview below); for student Q, it was his big brother who stimulated his interest in thinking about challenging problems; and for student R, it was her father who exposed her to thought-provoking and challenging problems as she was growing up. Like the exceptional group 1 and group 2 students, all group 3 interview subjects report that they value teachers who ask challenging questions and encourage independent thinking. Several also indicated that they appreciated teachers who made them feel comfortable in asking questions. Four of the five subjects indicated that their problem-solving skills and mathematical habits were acquired prior to entering the university, and their math instruction at the university level had not noticeably changed their approaches to doing mathematics. Three interview subjects had cautiously re-enrolled in a course they had already taken, in order to recall information or gain better understanding. This created an opportunity to genuinely understand the concepts rather than superficially learn what was necessary to acquire a grade. For student T, retaking first-semester calculus provided the first high grade he had ever received in a math course. Subsequently, he expected mathematics to make sense. Several group 3 interview subjects also indicated that, on at least one occasion, they had received an undeserved A, as they believed they had little understanding of the material taught in the course. This statement was in tandem with a description of this course and an admission that, during this course, memorization had been frequently substituted for the pursuit of understanding. All group 3 interview subjects appeared to possess excellent study habits and exceptional persistence when attempting to find solutions to challenging problems. This last trait appears to be the most important and distinguishing quality of these subjects. Since this is a cross-sectional investigation, it cannot be assumed that these beliefs and habits were acquired between the completion of second semester calculus and the completion of first-year graduate mathematics.

Mathematical background of Student U—Group 3.

Int: Tell me about your mathematical background, classes and teachers.

U: I went to a good high school in Northern Virginia, and I had one really excellent math teacher during my last year of high school and the others before that were pretty good, mostly.

Int: What class did you take your senior year?

U: It was called Math Analysis. It was a combination of algebra and trigonometry.

Int: Why did you think your teacher for that class was so outstanding?

U: She inspired us to think and made me feel like I could come up with solutions on my own. So it was more of an adventure than just memorization of information like my other math classes.

Int: Did she give you problems that stretched your abilities?

U: I don't know if it was so much the problems, but the way she taught it was she would allow us to think and offer our opinions by asking questions like, what do you think about this? It was one class in which I stayed awake. The whole class was very competitive and that's when I really became interested in math. Actually I didn't even think I could do well in math until I had this class. I had very bad math experiences before that class.

Int: Why were your earlier experiences so bad?

U: Early math was really hard like arithmetic, can you believe it? I don't know, it wasn't obvious that it had to work that way. We just did a lot of computational things and it just didn't make much sense. I always fell behind and wasn't really motivated to work. It wasn't until my senior math class that I realized math is pretty interesting.

Int: Where did you build your problem solving skills?

U: I think it was that class. I never knew how to read a math text before that class and I never knew how to study for a test until then. She taught us how to sort things out and told us to try to work the problem and then go back and figure out what things went wrong if we didn't totally get it. Never before that class did I go back and read my own work. I think now I could pick up a math text and teach myself just because of that class.

Int: What class did you take as a freshman in college?

U: I started in business calculus, because I didn't think I wanted to do five hours of math a day. I've always had a lot of interests and at that point really didn't think I wanted to go into math.

Int: What did you take after business calculus?

U: I guess I liked calculus enough that I went on and took the math calculus sequence, but they weren't required since I was a Spanish major. I had a great 122 (second-semester calculus) teacher (a TA).

Int: When did you decide to be a math major?

U: About a year and a half ago.

Int: You've had a lot of math in a very short amount of time.

U: Yes, after I came back from Costa Rica, I decided I really wanted to be a math major. Math is all I really want to do, but I really didn't think I could do it.

Int: So what changed your mind?

U: After I took set theory then I decided I really wanted to continue and I took several more courses after that.

Int: So how many classes are you taking this semester?

U: Both 791 (abstract algebra) and 765 (complex analysis).

Int: Have you had any post-secondary experiences that affected your mathematical habits?

U: I can't say that any professor has helped me to be a better problem solver, and I think that most of my mathematical habits were developed by my high school teacher. My high school math teacher made you feel every question is valid in class and she didn't put you down. Just the fact that a question is valid creates a whole attitude that every question should be valid and that's how it should be. It allowed you to search until you really understand. I've never had professors inspire me in the same way as my high school math teacher did.

Discussion

Gaining an understanding of the many components of the function concept is complex. It requires acquisition of a language for talking about its many features and the ability to translate that language into several different representations. Once students possess the skills for translating between these representations, they must learn to interpret features of each representation for many different types of functions. Then we ask that they recognize the usefulness of each representation for many different types of functions. Concurrently they are expected, on demand, to demonstrate the ability to construct each representation for a variety of real world situations. To further complicate matters, we ask that they learn a formal definition, somewhat inconsistent with the ways in which they use functions, and expect them to precisely apply this definition in arbitrary situations. At the same time, a process view of functions must emerge for understanding to become complete. Even this daunting scenario is no doubt an over-simplification of what really takes place as an individual struggles to make sense of functions. However, it suggests that understanding and assimilating the many aspects of functions requires a great deal of "sense making" on the part of the student.

Results indicate that as students progress through the undergraduate mathematics curriculum, function constructs develop slowly. Although students are eventually able to use concepts taught previously, even for our best students, complex concepts are slow to develop and new information is not immediately accessible. This was evident by identifying the collection of questions for which each group performed well and the collection of items for which each group performed poorly (see Appendix A and Appendix B). College algebra and second-semester calculus students had difficulty with the items assessing aspects of the function concept which were central to their respective courses. Despite these difficulties, second-semester calculus students performed very well on items measuring aspects of functions taught in early algebra, and graduate students demonstrated little difficulty with items measuring aspects of functions taught in beginning calculus. These results show that even the highest performing college algebra and second-semester calculus students demonstrate difficulties with much of the information explicitly taught during their courses. Interview results suggest that only after repeated access of a concept, in ways which require one to explore extreme cases and continue learning more about the various uses of the concept, does that knowledge become immediately accessible when confronted with a problem.

The successful college algebra students in this study had limited understanding of many of the components of the function concept. Their narrow view of functions was demonstrated by the fact that they thought any function could be defined by a single formula and that all functions must be continuous. They did not understand the function notation and had difficulty understanding the role of the independent and dependent variable given a functional relationship.

They could not explain what is meant when asked to express one quantity as a function of another, and were unable to verbalize the meaning of $f(x+a)$ given a quadratic function f. They were unable to speak the language of functions and during the interviews had difficulty referencing algebraic symbols. Although group 1 students were able to interpret points on a graph, they had difficulty interpreting graphical function information over intervals. They were able to construct the graphs of simple algebraic functions and algebraically evaluate functions for specific input values. According to Monk's [14] classification, group 1 students appeared to possess a pointwise view of functions.

Furthermore, the analysis of group 1 interview results reveals that college algebra students view the evaluation of a function as nothing more than an algebraic substitution. They speak of the function input as an item to be substituted and, as revealed in the interviews, have memorized the process. Group 1 students did not appear to distinguish between solutions of an equation and zeros of a function. Their immersion in an equation-oriented curriculum appeared to compromise their understanding of functions. They did not yet view functions more generally as a process which accepts input and produces output. In the Breidenbach et al. [3] classification, group 1 students appeared to possess an action level conceptualization of functions.

The high performing second-semester calculus students possessed a much broader view of functions. They demonstrated modest understanding of the language of functions, and had no difficulty when prompted to describe a physical situation using functions. They clearly understood what it means for one item to be a function of another and in most situations viewed functions more generally as processes. Although they had no difficulty interpreting static graphical information and very little difficulty interpreting dynamic graphical information, in certain situations they interpreted the graphical information literally rather than relying on their ability to interpret functional information displayed by the graph. Their understanding of function notation was accurate for simple algebraic function representations. However, when confronted with a more complex function expression they demonstrated some difficulties. When asked during the interview to determine graphically what was represented by the expression $F(\frac{x+y}{2})$, given a quadratic function F, most group 2 students had difficulty. In addition, they were unable to speak about $\frac{x+y}{2}$ as the input to this function. As well, the high-performing second-semester calculus students demonstrated surprising difficulties when attempting to solve problems requiring the use of beginning calculus. They also believed that functions are defined by a relationship between the items of two sets which is, in principle at least, predictable. Like group 1 students, some group 2 students still had difficulty distinguishing between an equation and a function and sometimes used the term "zero" synonymously with "solution." As a group they did not demonstrate confidence in their problem-solving abilities, nor did they access recently acquired calculus tools when solving unfamiliar problems. Most group 2 subjects were unable to interpret and graph covariant aspects of a real world situation. When attempting to graph height as a function of the volume of a spherical bottle filling with water, 74% of group 2 interview subjects constructed a graph which was either strictly concave down or concave up for the portion of the graph corresponding to the entire spherical part of the bottle. Even though they performed much better than group 1 students when asked to construct discontinuous functions, as a group they continued to demonstrate some difficulties defining piecewise functions. Although second-semester calculus students had begun to demonstrate a more general view of functions, when confronted with more demanding problems, they had a tendency to regress to lower level skills and, at times, demonstrate weak understanding. According to the Breidenbach et al. model [2], group 2 students appeared to be in transition from an action view to a process view of functions.

The beginning graduate students possessed a very general view of functions and had acquired attitudes, beliefs, and approaches of a maturity not possessed by members of the lower two groups. Although their motivations for studying mathematics were diverse, they all possessed perseverance, confidence and a willingness to work extremely hard in solving challenging and unfamiliar problems. They believed they were responsible for making sense of new information and indicated that at various points in their lives they assumed this responsibility personally.

The graduate students had no difficulty using the language of functions. Like the second-semester calculus students, they spoke about functions in terms of independent-dependent relationships, and viewed functions more generally as processes which accept different input values. Additionally, their general view is reflected by the fact that they accept the function definition, and unlike group 2 interview subjects, had no difficulty recognizing how an arbitrary mapping between two sets could determine a function. When required to determine a piecewise function, group 3 interview subjects all provided a correct response, indicating no difficulty describing different rules for different parts of the domain. When asked to interpret graphical information for ranges of the domain, group 3 students, like group 2, had no difficulty providing a correct response and valid justification. On items which could more easily be solved using tools of calculus (e.g. analyzing the relative areas under the curve, examining the slope of the tangent line, analyzing how changing rate affects concavity, etc.), group 3 students more frequently accessed their knowledge of calculus to justify their response. In addition, group 3 students demonstrated a much greater tendency to interpret covariant aspects of a situation as demonstrated by their analysis of the "bottle filling with water" problem, and the "rate of change of temperature" problem [3]. When required to analyze and compare complicated expressions for a quadratic function, group 3 students demonstrated very little difficulty. All group 3 interview subjects discussed $F(\frac{x+y}{2})$ as a process of averaging the input values, followed by an evaluation of the function for that value;

and $\frac{F(x)+F(y)}{2}$ as evaluating the function for two different inputs, followed by averaging the two functional values. Group 3 subjects gave graphical representations for each of these expressions with very little difficulty and were able to compare the expressions for different quadratics. In general when group 3 students demonstrated difficulty they were able to correctly analyze the situation with very little prompting and provide a correct response.

Collectively, the interview results suggest that students acquire concepts at very different rates. Though each group consisted of A students, dramatic differences were noticed among the interview subjects of each of the two lower groups. In fact, two of the second-semester calculus interview subjects frequently demonstrated understanding at the same level as the graduate students, as illustrated by their high scores on the written exam, and their exceptional ability to reason through difficult problems during their interviews. College algebra interview subjects' understandings of major function components had a range from very weak to exceptional.

A continual effort of sense making occurs as students progress through undergraduate mathematics. However, the rapid pace at which new information is presented eliminates needed time for reflection and appears to encourage students to settle for superficial understanding. It is likely that many capable students prematurely discontinue their study of mathematics due to their unwillingness to substitute understanding with memorization. In fact, the student who demonstrated the greatest understanding among the group 1 interview subjects indicated that this was the reason she had decided to pursue a Ph.D. in English, versus continuing her study of mathematics.

As students struggle to understand new information presented at a rapid pace, it is difficult to predict what will be retained by a particular student, since individuals appear to be willing to devote varying amounts of effort to sort out information. Additionally, I suspect that different levels of intelligence and memory capabilities also affect their ability to understand newly presented information at a fast rate. However, this research suggests that even the very best students frequently abandon efforts to understand new concepts when those concepts are presented at a rate which does not allow time for reflection and exploration of special cases. This raises questions concerning the level of understanding that occurs in average and below-average mathematics students.

Despite these difficulties, some students do survive undergraduate curricula and eventually acquire excellent mathematical habits and abilities. Group 3 subjects, students who successfully completed undergraduate mathematics degrees, demonstrated understanding of major aspects of functions and possessed exceptional problem-solving strategies. They reported specific experiences which fostered the development of their mathematical habits (see Backgrounds and Beliefs of Group 3 Interview Subjects), and indicated that concept development was facilitated by a variety of factors: solving challenging problems, repeated exposure to essential concepts, repeating a course which was already taken, accessing the same information when taking a higher-level course, or teaching a course at a later point in their development. Further, results suggest that concept development appears to be facilitated by engaging in experiences that encourage the acquisition and assimilation of many sub-concepts. Group 3 students also exhibited exceptional persistence and confidence when responding to complex tasks during the interview. In contrast, group 1 and 2 interview subjects were reluctant to persist and demonstrated little or no confidence in their mathematical abilities. Consequently, good mathematical habits (e.g., persistence, problem-solving strategies, etc.) and confidence, in addition to concept development, appear to be essential for students to continue their study of mathematics.

Conclusions and Implications

The following are general conclusions reached as a result of this research.

1. Function constructs develop slowly and their development appears to be facilitated by reflection and constructive activities. Interview results suggest that when students are confronted with engaging activities and provided time for reflection, student understanding is promoted. These results agree with those of Breidenbach et al. [2]. Their study found that understanding of functions was noticeably improved as a result of engaging students in constructive activities.

2. Even the most talented students at the completion of college algebra still have many misconceptions. Analysis of the exam responses and interview transcripts for group 1 indicates that these students:

   - Do not understand the language of functions, in particular:
     What it means for one quantity to be a function of another.
     The role of the parentheses in the function representation.
     That the "functional value" is referring to the y-value
     (assuming conventional labeling of the axes).
   - Do not know how to represent real world functional relationships using algebraic or graphic function representations.
   - Do not make a distinction between zeros of functions and solutions of equations.
   - Do not effectively interpret dynamic graphical information.
   - Do not understand the general nature of a function. They think any function must be definable by a single algebraic formula and all functions must be continuous. This result is consistent with the results of Ayers et al. [1] and Vinner and Dreyfus [21].
   - Do not understand the role of the independent and dependent variables in an algebraic function representation.

- Do not represent graphically, covariant aspects of a real world situation.
- Do not possess helpful mathematical habits. They do not view their mathematics as useful for solving problems, do not expect mathematics problems to "make sense," and have little confidence in their abilities to solve unfamiliar problems.

3. The most talented second-semester calculus students, at the completion of the course, demonstrate some difficulty:
   - Interpreting rate of change information from a dynamic situation.
   - Demonstrating an awareness of the impact change in one variable has on the other; interpreting and graphically representing covariant aspects of a real-world situation.
   - Accessing beginning calculus to analyze a real world situation.
   - Defining a discontinuous function using a different rule for different parts of the domain. Group 2 students still think that any function must be definable by a single algebraic formula.
   - Interpreting dynamic graphical information over intervals.
   - Understanding function notation for function representations.
   - Translating from complicated algebraic to graphic representations.
   - Using tools of calculus to analyze a dynamic situation.
4. High-performing students do not appear to access recently acquired function information to solve unfamiliar problems (see Appendix B), and do not demonstrate belief in their mathematical knowledge as useful when they are required to solve an unfamiliar problem, even when it is solvable using the mathematics they know. This result is consistent with the findings of Schoenfeld [17]. When investigating students' abilities to apply known geometric information to geometric construction problems, Schoenfeld [17] found that successful mathematics students gave evidence of knowing certain mathematics but then proceeded to ignore the information when attempting the constructions.
5. It appears that curriculum developers underestimate the complexity of acquiring an understanding of many of the essential components of the function concept. Many of the weaknesses identified in group 1 and group 2 students are not specifically addressed in current curricula. For example, current curricula provide little opportunity for developing the ability to: interpret and represent covariant aspects of functions, understand and interpret the language of functions, interpret information from dynamic functional events, etc.
6. Students report that they replace understanding with memorization in the absence of time for reflection, questioning, and exploration of extreme cases

and special situations. All group 3 (graduate level) interview subjects and most group 2 (second-semester calculus) interview subjects indicated that the rapid pace of a particular class had led to frustration and the abandonment of understanding for memorization.

7. In order to develop good mathematical habits, good students believe they need to be challenged by working more difficult problems. All interview subjects in group 3 and most of the interview subjects in group 2 indicated that their approach to doing mathematics had been acquired when they were encouraged, challenged, and given guidance regarding different strategies for solving what seemed like a difficult problem (see Backgrounds and Beliefs section).

8. Good students report that they want mathematics to "make sense," prefer mathematics to be taught in context, and like being challenged by difficult problems. They report that they do not like monotonous, repetitious activities, rather they enjoy engaging in rich mathematics that is interesting and purposeful. They praise their teachers for providing challenging problems while guiding their solution attempts and creating a non-threatening classroom environment.

9. Full concept development appears to evolve over a period of years and appears to require an effort of "sense making" to understand and orchestrate individual function components to work in concert [3].

This investigation provides empirical evidence for the importance of constructive activities in the development of one concept. Further, this study has identified essential aspects of the function concept which need increased attention, while providing insights concerning the types of experiences and curricula which may foster their development. The pace at which content is presented, the context in which it is presented, as well as the types of activities in which we engage students appear to have an enormous impact on what students know and what they can do when they exit a course. Consequently, curriculum developers have a tremendous responsibility to gain as much information as is currently available describing how students acquire the concepts specific to a course, as well as mathematical concepts in general.

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APPENDIX A
The Written Exam

Student Background Information

On Parts A and B of the Functionality Test, students were asked to supply the following background information:

Name (optional)
Level in School
Last math course completed
Semester last math course was completed
Math courses completed
Grade expected in current math course (Part A only)

Functionality Test—Part A

1. Express the diameter of a circle as a function of its area and *sketch its graph.*

2. a) Given $f(x) = 2x+3$, what is the relationship between $f(x+1)$ and $f(x)+2$ for the given function?
   Explain.

   b) Find k so that $g(x+1) = g(x) + k$, given that $g(x) = 3x + 5$.
   Explain.

   c) Compute $h(x+a)$ given $h(x) = 2x + 3$.

   d) Compute $f(x+a)$ given $f(x) = 3x^2 + 2x - 4$

3. If possible, describe the following situations using a function. If not, explain why.

   a) The string, "ABCDEFG"

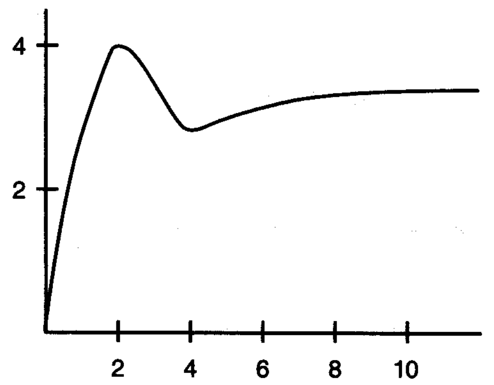
   b) $\{2n + n^3 : n \text{ in } [1..100]\}$

   c) $y^4 = x^2$

   d) The club members' dues status.

Name	*Owed*
Sue	$17
John	$6
Sam	$27
Bill	$0
Iris	$6
Eve	$12
Henry	$14
Louis	$6
Jane	$12

4. This graph shows the speed in meters per second of a cyclist over a 10 minute period.



a) When is the speed greatest?

b) When is the speed changing most rapidly?

c) Does the cyclist travel further during the first five minutes or during the last five minutes?

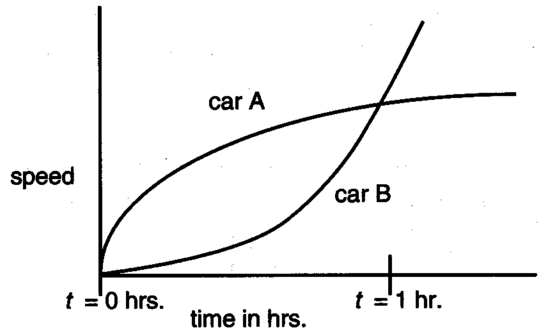
In questions 5 through 7, give an example to confirm the existence of such a function. If one does not exist, explain why.

5. Does there exist a function all of whose values are equal to each other?

6. Does there exist a function whose values for integer numbers are non-integer and whose value for non-integer numbers are integer?

7. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

8. The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.)



a) State the relationship between the position of car A and car B at $t = 1$ hr.: Explain.

b) State the relationship between the speed of car A and car B at $t = 1$ hr.: Explain.

c) State the relationship between the acceleration of car A and car B at $t = 1$ hr.: Explain.

d) What is the relative position of the two cars during the time interval between $t = .75$ hr. and $t = 1$ hr.? (i.e. is one car pulling away from the other?) Explain.

9. The table on the left represents specific values of the function $f(x) = x^3 - 3x^2 + 2x$. Fill in the table on the right, which represents the function $g(x) = x^3 - 3x^2 + 2x + 1$.

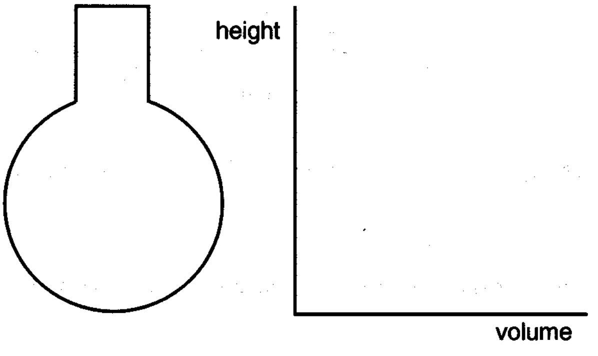
x	y
0	0
1	0
2	0
3	6
4	24

x	y
0	
1	
2	
3	
4	

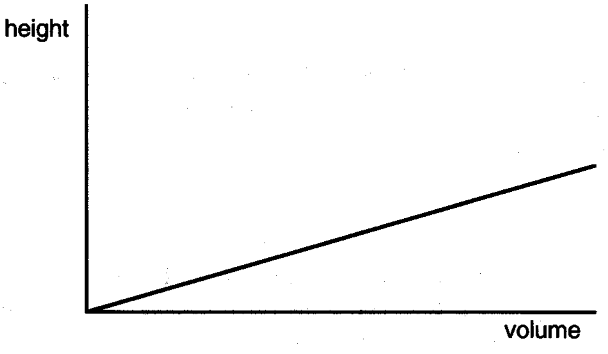
10. Sketch rough graphs of $f(x) = x^2 - 4$ and $g(x) = 3x$, and discuss the solution to the equation $f(x) = g(x)$ in terms of these graphs.

11. Sketch a rough graph of $h(x) = (x-4)(x+1)$ and discuss the relationship of the solution of the equation $h(x) = 0$ to the solution of the equation $f(x) = g(x)$ from 10 above.

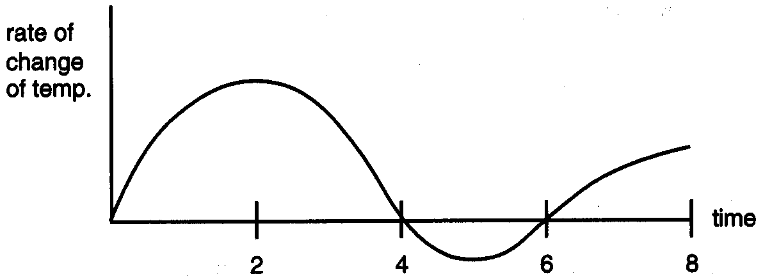
12. Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle.

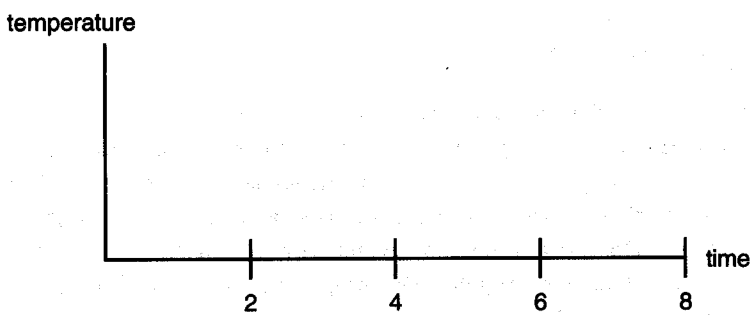


13. Suppose this is the graph of height as a function of volume as a bottle is filling with water. Sketch the shape of the bottle.



14. Given the graph of the rate of change of the temperature over an eight hour time period. Draw a rough sketch of the graph of the temperature over the same eight hour time period. Assume the temperature at time, $t = 0$, is 0 degrees Celsius.





Functionality Test—Part B

1. a) What is a function?

   b) Describe the different ways a function can be represented.

   c) What is the value of studying functions?

2. Assume $F(x)$ is any quadratic function.

   a) True or False: $F(\frac{x+y}{2}) < \frac{F(x)+F(y)}{2}$

   b) Justify your answer.

3. a) Find the equation of the line(s) through the point (a, a^2) that intersects the graph of $y = x^2$ exactly once.

   b) Explain your solution.

4. a) Tom sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount and then again by the same amount and then again by the same amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do the amounts by which the top of the ladder drops down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? EXPLAIN.

   b) Newt, the science nerd, then comes along and puts wheels on the bottom of the ladder. He connects them to a motor so that the bottom rolls away at a constant, but very slow, speed. Does the top of the ladder move down at a constant speed, or does it speed up or does it slow down? EXPLAIN.

   c) Draw a graph which represents the relationship between the horizontal and vertical positions of a ladder as it slides down a wall, starting at a vertical position and finally resting on the ground. EXPLAIN.

APPENDIX B
Quantitative Results
Part A—Written Exam

Question No.		Group 1 mean	Group 2 mean	Group 3 mean	$2 > 1$ $\alpha = .05$	$3 > 2$ $\alpha = .05$	$3 > 1$ $\alpha = .05$
**	1	1.00	3.29	4.83	yes	no	yes
**	2a	1.67	4.18	4.75	yes	no	yes
*	2b	1.53	3.71	4.75	yes	yes	yes
**	2c	2.43	4.94	5.00	yes	no	yes
**	2d	2.07	4.76	5.00	yes	no	yes
	3a	.80	1.71	4.58	no	yes	yes
*	3b	.47	3.35	4.67	yes	yes	yes
*	3c	1.07	2.71	4.50	yes	yes	yes
	3d	.70	.53	5.00	no	yes	yes
	4a	4.30	4.71	5.00	no	no	no
**	4b	2.60	4.12	5.00	yes	no	yes
*	4c	3.93	4.88	5.00	no	no	no
**	5	1.07	4.00	4.67	yes	no	yes
*	6	.17	1.00	5.00	yes	yes	yes
*	7	.50	3.18	4.75	yes	yes	yes
*	8a	.83	2.82	5.00	yes	yes	yes
**	8b	2.80	4.80	5.00	yes	no	yes
*	8c	2.10	3.35	5.00	yes	yes	yes
	8d	.43	1.18	3.83	no	yes	yes
	9	4.60	4.94	5.00	no	no	no
**	10	3.30	4.47	4.50	yes	no	yes
**	11	1.63	4.35	4.67	yes	no	yes
*	12	1.47	2.41	4.33	yes	yes	yes
	13	1.27	5.00	4.92	yes	no	yes
*	14	.33	2.59	4.92	yes	yes	yes

* Group 2 performed significantly higher than group 1 and group 3 performed significantly higher than group 2.

** Group 2 performed significantly higher than group 1 and group 3 did not perform significantly higher than group 2.

Quantitative Results
Part B—Written Exam

	Question No.	Group 1 mean	Group 2 mean	Group 3 mean	2 > 1 $\alpha = .05$	3 > 2 $\alpha = .05$	3 > 1 $\alpha = .05$
	1a	2.07	2.71	5.00	no	yes	yes
	1b	2.63	3.18	5.00	no	yes	yes
*	1c	1.67	3.53	5.00	yes	yes	yes
*	2	.10	1.35	4.33	yes	yes	yes
*	3	.50	2.06	3.33	yes	yes	yes
*	4a	.63	2.41	4.33	yes	yes	yes
*	4b	.43	2.24	4.08	yes	yes	yes
	4c	.63	2.06	4.33	yes	yes	yes

* Group 2 performed significantly higher than group 1 and group 3 performed significantly higher than group 2.
** Group 2 performed significantly higher than group 1 and group 3 did not perform significantly higher than group 2.

APPENDIX C
Scoring Rubric Guidelines

The following general guidelines were used for constructing a rubric for each exam question.

General Evaluation

Question
Scoring

5 Superior response:
- -Complete in responding to all aspects of the question.
- -Shows complete mathematical understanding of the problem's ideas and requirements.
- -Includes only minor computational errors, if any.

4 Assign to those responses falling between 5 and 3.

3 Adequate response:
- -Demonstrates understanding of the main idea of the problem.
- -Is not totally complete in responding to all aspects of the problem.
- -Shows some deficiencies in understanding aspects of the problem.
- -Exhibits a moderate amount of reasoning but reasoning is incomplete.

2 Assign to those responses falling between 3 and 1.

1 An inadequate response:
- -Attempts, but fails to answer or complete the question.
- -Shows very limited or no understanding of the problem.
- -Contains words, examples, or diagrams that do not reflect the problem.

0 No response:
- -The question was left blank.
- -The written information made no attempt to respond to the problem.
- -The written information was insufficient to allow any judgment.

Rubrics for Specific Problems

Part A

Question 1

(3 pts. for algebraic solution and 2 pts. for graph.)

1 pt. Writes formula for the area of a circle.

2 pts. Substitutes $(d/2)$ for r.

3 pts. Solves for d, $d = 2\sqrt{A/\pi}$, and recognizes that the range is restricted, diam. is always positive. (Deduct 1 pt. for including "+" and "−".)

1 pt. Constructs graph, but graph has errors, i.e. both the positive and negative square root are graphed.

2 pts. Correctly sketches the graph. (Note: A picture of a circle receives 0 pts. for graph.)

Question 2a

1 pt. Evaluates $f(x+1)$ or $f(x)+2$.
or
States that $f(x+1)$ is a left shift and $f(x)+2$ is an upward shift of $f(x)$.

2 pts. Evaluates $f(x+1)$ and $f(x)+2$.

3 pts. Sets $f(x+1)$ equal to $f(x)+2$.

4 pts. States that $f(x+1)$ is equal to $f(x)+2$.

5 pts. Valid explanation is provided.
States that $f(x+1)$ and $f(x)+2$ represent the same line and provides justification of their equality, i.e. discusses why adding 1 to the input results in adding 2 to the output.

Question 2b

1 pt. Evaluates $g(x+1)$ or $g(x)+k$.

2 pts. Evaluates $g(x+1)$ and $g(x)+k$.

3 pts. Sets $g(x+1)$ equal to $g(x)+k$ and determines that k is equal to 3.

4 pts. Weak explanation is provided.
An algebraic explanation is provided.

5 pts. Strong explanation is provided.
Provides a discussion concerning the relationship of k and the slope of g. Discusses how adding one to the input affects the output.

Question 2c

0 pts. Adds "a" to the output, $h(x+a) = (2x+3)+a$.

5 pts. Correctly evaluates $h(x+a)$.
$h(x+a) = 2(x+a)+3$
or
$h(x+a) = 2x+2a+3$.

Question 2d

1 pt. Partial attempt to replace x with $(x+a)$.
$f(x+a) = 3(x+a)^2 + 2x - 4$
or
$f(x+a) = 3x^2 + 2(x+a) - 4$.

4 pts. Error in simplifying 5 pt. answer.

5 pts. Correct evaluation of $f(x+a)$
$f(x+a) = 3(x+a)^2 + 2(x+a) - 4$
or
$f(x+a) = 3x^2 + 6xa + 3a^2 + 2x + 2a - 4$

Question 3a

4 pts. Minor errors in 5 pt. answer.

5 pts. Not possible because there is only one set and two sets are needed to define a function.

5 pts. A valid association of a selected input with each letter of the string.

5 pts. A valid association of the selected input with the entire string.
$f(1) = A,\ f(2) = B,\ f(3) = C,\ f(4) = D \ldots f(7) = G$
or $f(a) = A,\ f(b) = B,\ f(c) = C,\ f(d) = D \ldots f(g) = G$
or $f(5) =$ABCDEFG
or defines the string as a point, which maps the string to itself,
$f($ABCDEFG$) =$ ABCDEFG.

Question 3b

2 pts. Statement that the given representation is a function.

3 pts. A valid explanation which represents understanding of the functional nature of the situation, but no function is defined.

4 pts. $f(n) = 2n + n^3$, error in domain or no domain is stated.

5 pts. $f(n) = 2n + n^3$, n is an element of the integers and $1 \leq n \leq 100$.

Question 3c

2 pts. A valid function representation which restricts the domain and range of the function $y = \sqrt{x}$.

3 pts. A valid function representation which restricts the domain or the range: $y = (+ \text{ or } -)\sqrt{x}$
or
$y = \sqrt[4]{x^2}$.

4 pts. All points are represented by the equation, but no indication that the student recognizes that this is not a function. $y = (+ \text{ or } -)\ \sqrt{|x|}$.

5 pts. Above answer with qualification that the equation is not a function.

5 pts. Not possible to describe the set of points with a function, since any attempt will violate the function restriction.

Question 3d

1 pts. Statement that a pairing is possible, but no pairing is constructed.

2 pts. Pairing which associates the two sets. However, the pairing is not a function, i.e. $f(17) =$ Sue, $f(6) =$ John, $f(6) =$ Iris.
or
Not possible since the function is not one-to-one.
(Student recognizes a pairing exists, but thinks all functions must be one-to-one.)

5 pts. Pairing the two sets in such a way that a function is defined.
Use of arrows to make association
or (Sue, 17), (John, 6),...
or f(Sue) $= 17$, f(John) $= 6$, ...

Question 4a

1 pt. Student writes a "4."

2 pts. Student writes "4 meters/second."

5 pts. Any answer indicating the speed is the greatest when $t = 2$ minutes.
At $t = 2$ minutes or at $t = 2$ minutes when speed $= 4$ mph or (2,4).

Question 4b

1 pt. $t = 2$ min. to $t = 4$ min.
or
$t = 3$ min. to $t = 4$ min.

2 pts. $t = 2.5$ minutes
or
Any subset of the interval from $t = 1$ to $t = 2$ minutes.

3 pts. $t \sim .5$ minutes.

5 pts. $t = 0$ to $t = 2$ minutes.
or
$t = 0$ to $t = 1$ minutes.

Question 4c

0 pts. First 5 minutes.

5 pts. Last 5 minutes.

Question 5

1 pt. $f(x) = x$.

3 pts. Writes a point, not in set notation (i.e. (3,5)).

5 pts. A function meeting the condition that all function values are equal to each other (i.e. $f(x) = c$ or $\{(3,5)\}$).

Question 6

1 pt. Responds with "yes." Provides no explanation or example.

2 pts. Function satisfies one of the criteria-"value for integer values are non-integer" or "value for non-integer values are integer," i.e., $f(x) = \frac{1}{x}$ or $f(x) = [x]$.

4 pts. Minor errors in 5 pt. answer.

5 pts. Valid function definition using a split domain.
$f(x) = 5$ if x is not an integer.
$f(x) = \frac{1}{x}$ if x is an integer.

Question 7

1 pt. Responds with "yes." Provides no explanation or example.
or
Constructs a function such that $f(0) = 1$ (i.e. $f(x) = c^x$).

2 pts. A valid graph, but responds with "no" and states as a reason that the graph is not continuous. (Student thinks all functions must be continuous.)

4 pts. Minor errors in 5 pt. answer.

5 pts. $f(x) = x^2$ if x is not equal to 0
$f(x) = 1$ if $x = 0$.

Question 8a

0 pts. States that the cars have traveled the same distance.

1 pt. States that car A or car B is accelerating.

2 pts. States that car A and car B are both accelerating.

3 pts. Correct statement of position with no explanation.
Car A is ahead of car B.

4 pts. Correct statement of position with weak justification.
The graph of car A is higher.

5 pts. Correct statement of position with valid explanation.
Car A is ahead of car B, since car A has been going faster the entire time.
or
Car A is ahead of car B, since the area under the graph of car A is greater than the area under the graph of car B.

Question 8b

3 pts. Correct statement of the speed of the two cars at $t = 1$ hr.
The cars are going the same speed at $t = 1$ hr.

4 pts. Correct statement of the speed with weak justification.
The graphs intersect.

5 pts. Correct statement of speed with strong justification.
The cars are going the same speed since their graphs intersect at $t = 1$.
or
The cars are going the same speed since at $t = 1$, the speeds are the same.

Question 8c

3 pts. Correct statement of the relative accelerations at $t = 1$ hr.
The acceleration of car B is greater than the acceleration of car A at $t = 1$ hr.

4 pts. Correct statement with weak justification.
Car B's speed appears to be increasing faster.

5 pts. Correct statement of relative accelerations and valid explanation.
Acceleration of car B is greater than the acceleration of car A, since the graph of car B is steeper than the graph of car A at $t = 1$ hr.
or
Acceleration of car B is greater than the acceleration of car A, since the slope of the tangent line of the graph of car B is greater than the slope of the tangent line of the graph of car A at $t = 1$ hr.

Question 8d

0 pts. Car B is catching up with car A. No explanation.

1 pt. Car B is gaining on car A, because car B's acceleration is greater than Car A's acceleration.

3 pts. Correct statement of the cars' relative positions between $t = .75$ hr. and $t = 1$ hr. Car A is pulling away from car B.

5 pts. Correct statement of the cars' relative positions with a valid explanation.
Car A is traveling faster than car B during the entire time interval from $t = .75$ to $t = 1$ hr., so car A is pulling away.

Question 9

1 pt. awarded for each ordered pair.
(Method used to obtain the table entries is noted.)

Question 10

1 pt. Correct graph of f or g.

2 pts. Correct graph of f and g.

3 pts. Statement that the solutions are the pts. where the graphs intersect.

4 pts. Identification of the two points as the solution of the equation.

5 pts. Identification of the x values of the points of intersection as the solution to the equation (i.e. $x = 4$ and $x = -1$).

Question 11

1 pt. Correct sketch of the graph of h.

2 pts. States the solution is the points where h crosses the x-axis.
or
Identifies the two points, $(-1, 0)$ and $(4, 0)$, as the solutions to $h(x) = 0$.

3 pts. Identifies the x-values where h crosses the x-axis as the solution to $h(x) = 0$, $x = -1$, $x = 4$.

4 pts. States that the solutions of $h(x) = 0$ are also the pts. of intersection of $f(x)$ and $g(x)$.

5 pts. States that the solutions of $h(x) = 0$ are the same as the solutions to $f(x) = g(x)$ from 10 above, $x = -1$, $x = 4$.
or
States that the observed relationship is that $h(x) = f(x) - g(x)$.

Question 12

Award 1 pt. for correct display of each of the following graphical features.

1 pt. The graph of the function is always increasing.

1 pt. Graph shows a change in concavity and nonlinear portion is symmetric about the concavity change.

1 pt. Graph changes from concave down to concave up.

1 pt. Graph becomes linear.

1 pt. Slope of the linear portion of the graph is the same as the slope of the tangent line of the last point of the curve before becoming linear.

Question 13

0 pts. Any figure other than the two described below.

2 pts. Draws a cylindrical shaped figure with neck.

5 pts. Draws a cylinder.

Question 14

0 pts. Draws the same graph which is given.

Award 1 pt. for correct display of each of the following graphical features.

1 pt. The graph is increasing from $t = 0$ to $t = 4$.

1 pt. The graph is decreasing from $t = 4$ to $t = 6$.

1 pt. The graph is increasing from $t = 6$ to $t = 8$.

1 pt. The graph changes from concave up to concave down at $t = 2$.

1 pt. The graph changes from concave down to concave up at $t = 5$.

Part B

Question 1a

1 pt. An example of one function representation, equation, graph, correspondence or an equation with an independent and dependent variable.

2 pts. A graph which passes the vertical line test.

3 pts. Provides a general description without reference to the function restriction, i.e. a relation between two sets.

4 pts. Minor error in 5 pt. answer.

5 pts. A relation which maps each element of one set to exactly one element of another set or another correct definition.

Question 1b

1 pt. Provides example(s) of function(s).

2 pts. Describes one way of representing a function, i.e. an equation or a graph.

3 pts. States two forms of algebraic function representations. Explicitly defined in terms of y and implicitly defined in terms of y.

4 pts. An equation and a graph.

5 pts. Includes another valid function representation, i.e. table, verbal, Venn diagram, written description, set of pts.

Question 1c

3 pts. To learn more math.

5 pts. Answer demonstrates that studying functions has some personal value to the student, i.e. to describe real world relationships mathematically.

Question 2

0 pts. Student responds with "true" or "false" with no explanation.

1 pt. $F(\frac{x+y}{2})$ or $\dfrac{F(x)+F(y)}{2}$ is represented algebraically or geometrically.

2 pts. Both $F(\frac{x+y}{2})$ and $\dfrac{F(x)+F(y)}{2}$ are represented algebraically or geometrically.

3 pts. Weak attempt to compare $F(\frac{x+y}{2})$ and $\dfrac{F(x)+F(y)}{2}$ algebraically or geometrically.

4 pts. Same as 5 pt. solution with minor errors.

5 pts. Algebraic solution demonstrating that $F(\frac{x+y}{2}) < \dfrac{F(x)+F(y)}{2}$ when $\alpha < 0$.

or

Geometric argument demonstrating that $F(\frac{x+y}{2}) < \dfrac{F(x)+F(y)}{2}$ when the parabola opens downward.

or

Answer of "False" with a valid counterexample.

False when $x = y = 0$, since $F(\frac{x+y}{2}) = \dfrac{F(x) + F(y)}{2}$ when $x = y = 0$.

Question 3

(3 pts. awarded to tangent line and 2 pts. awarded to vertical line.)

1 pt. Computes $y' = 2x$.

or

Determines a specific tangent and/or vertical line at a point (i.e. $x = 0$ at $(0, 0)$ or $y = 0$ at $(0, 0)$).

2 pts. Determines that the slope of the line is 2a

or

Draws the graph of a quadratic, labels the pt. (a, a^2) and draws a tangent line through the pt.

3 pts. Writes the equation of the tangent line.

$$y - a^2 = 2a(x - a)$$

2 pts. Writes the equation of the vertical line through a, $x = a$.

Question 4a

1 pt. Larger, no justification.

2 pts. Larger, weak justification (i.e. the angle between the ladder and the wall increases, because of Pythagorean theorem, etc.).

or

Writes Pythagorean relationship, but states that the change remains constant.

3 pts. Larger, argues by use of a physical model.

Draws successive pictures of the base of the ladder being pulled away from the wall.

4 pts. Minor errors or incomplete 5 pt. answer.

5 pts. Larger, strong mathematical justification.

A complete algebraic solution is provided with discussion of why $\dfrac{dy}{dx}$ must be getting larger.

or

A strong numerical argument is provided by substituting values into the Pythagorean relationship.

or

A graph of ht. as a function of distance from the wall is provided along with the argument that the slope of the tangent line becomes steeper (larger) as the ladder gets closer to the floor (ht. gets closer to 0), so $\dfrac{dy}{dx}$ gets bigger.

or

A valid trigonometric solution is provided.

Question 4b

1 pt. Speeds up with no justification.

2 pts. Speeds up with weak justification.
or
Pythagorean relationship is provided, but response is incorrect, i.e. slows down or stays the same.

3 pts. Speeds up with informal practical argument.

4 pts. Speeds up and minor errors in 5 pt. argument or uses results from 4a to make an informal argument.

5 pts. Speeds up with strong mathematical justification.
Computes $\frac{dy}{dt}$ and argues that the speed gets larger as x, the distance from the wall, increases and $\frac{dx}{dt}$ is held constant.

Question 4c

1 pt. Provides Pythagorean relationship, $x^2 + y^2 = z^2$.
or
Provides picture of ladder sliding down the wall.

2 pts. Solves Pythagorean relationship for one of the variables and no graph is provided or
Graph is incorrect (i.e. $y = \sqrt{z^2 - x^2}$).

4 pts. Same as 5 pt. answer w/o correct labeling of the x and y-intercepts.

5 pts. Provides complete graph of functional relationship with correct labeling of the x and y-intercepts.
(Graph is upper rt. quarter of a circle with radius = length of ladder.)
Note: 0 pts. are to be awarded for a function which looks like the graph of $f(x) = \frac{1}{x}$ in the first quadrant.

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